

Inhomogeneous cosmologies, the Copernican principle and the cosmic microwave background: More on the EGS theorem

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Abstract

We discuss inhomogeneous cosmological models which satisfy the Copernican principle. We construct some inhomogeneous cosmological models starting from the ansatz that all the observers in the models view an isotropic cosmic microwave background. We discuss multi-fluid models, and illustrate how more general inhomogeneous models may be derived, both in General Relativity and in scalar-tensor theories of gravity. Thus we illustrate that the cosmological principle, the assumption that the Universe we live in is spatially homogeneous, does not necessarily follow from the Copernican principle and the high isotropy of the cosmic microwave background. We also present some new conformally flat two-fluid solutions of Einstein’s field equations.

1 Introduction

The standard model of cosmology rests on several fundamental assumptions. As with any theoretical model of a physical system, it is crucial that these assumptions are identified and tested wherever possible, in order that the proposed model be considered acceptable. The standard model of cosmology is founded on the (perturbed) spatially homogeneous and isotropic cosmological models of Friedman-Lemaître-Robertson-Walker (FLRW), which are derived from the cosmological principle. The cosmological principle may be taken to state that the universe is spatially homogeneous [1]. This is a strong assumption; considerably stronger than the Copernican principle which says that we are not at a special location in the Universe. Regardless of which principle one cares to take, when these are combined with *assumed* perfect isotropy about ourselves, on all scales, we arrive at the FLRW models. If we assume such perfect isotropy about us

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without a ‘mediocrity’ principle, then we must be at a center of symmetry. Obviously, the properties of a spatially homogeneous universe can be radically different from a spherically symmetric one, for example, and it is therefore of fundamental importance to find some method to test the assumption of homogeneity, and to identify what exactly happens if any of the assumptions are relaxed. This is particularly important, since studying inhomogeneous models will allow us to identify some possible tests of non-homogeneity in the universe.

What evidence do we have that the universe is so isotropic about us? Obviously the strongest and most important piece of evidence for this is the extremely high isotropy of the cosmic microwave background (CMB) which is isotropic to one part in 10^5 . The question is: In what context can we infer spatial homogeneity from our observations of the CMB? Without the Copernican principle (or something similar) the answer is obviously not – we may be located at the ‘center’ of the universe and thus see the CMB isotropically distributed about us, whereas all other observers would not have such a unique and privileged view. However, if we assume the Copernican principle (i.e., we assume that all or ‘most’ observers in the universe see the CMB to be as isotropic as we see it) can we infer homogeneity on the basis of the CMB alone?

The first attempt to answer this question resulted in a theorem by Ehlers, Geren and Sachs [2] (hereafter, EGS) which states that if all observers in an expanding dust universe see an isotropic radiation field then that spacetime is homogeneous and isotropic (and therefore FLRW). This can trivially be generalised to the case of a geodesic and barotropic perfect fluid [3]. The ‘isotropic radiation field’ is implicitly identified with the CMB. However, as has been emphasised recently [4, 5], the resulting spacetime will be FLRW only if the matter content is of perfect fluid form, and the observers geodesic and irrotational. This work has been extended [4] to include inhomogeneous universe models with non-geodesic observers. That is, inhomogeneous spacetimes have been found which also allow every observer to see an isotropic CMB. It has also been shown that a significant subset of these models are consistent with other observational constraints, regardless of observer position [6]. This means that these models are consistent with observations on all scales even when the Copernican principle is taken into account – and yet the models are significantly inhomogeneous. However, a problem of these models is that the non-geodesic motion of the observers prohibits a barotropic equation of state for the matter (although the models admit a thermodynamic scheme). There have been recent developments along this line, where in [7] a realistic multi-fluid form of the matter was proposed (although the models used were slightly different from those in [4] which allow an isotropic radiation field). Alternatively, a fractal distribution may be more appropriate [8]. Other recent work concerning rotating and anisotropic cosmological models also supports these results [9, 10, 11, 12]; other non-FLRW models exist which admit an isotropic radiation field.

The purpose of this paper is to discuss more general cosmological models which allow an isotropic radiation field. Specifically, we wish to discuss models with ‘realistic’ matter in order to demonstrate that there exist physically viable inhomogeneous cosmological models which will allow an isotropic radiation field by construction but are not FLRW. The recent supernovae data imply an accelerated expansion rate in the universe: within the standard model this implies some sort of negative pressure, be it a cosmological constant or quintessence or some other type of exotic matter. Therefore, we will consider here not just traditional barotropic perfect fluid matter, but more exotic forms, such as scalar fields and varying Λ models.

Spacetimes which allow all observers the view of an isotropic CMB must satisfy the ‘isotropic radiation field theorem’. The isotropic radiation field theorem may be derived from the Einstein-Boltzmann equations for photons in a curved spacetime. It is easy to show from the multipole expansions of [3] that a spacetime with an isotropic radiation field must have the velocity field, u^a , of the photons being shearfree and obeying

$$\dot{u}_a = \tilde{\nabla}_a Q, \quad \theta = 3\dot{Q}, \quad (1)$$

where Q is a function of the energy density of the radiation field. Any observers traveling on this congruence will observe the isotropic radiation. This velocity field is also a conformal Killing vector of the spacetime. In fact, a spacetime admitting an isotropic radiation field must be conformally stationary, and we use this fact to construct some ‘generalised-EGS’ spacetimes.

In the following we show how some irrotational multifluid spacetimes may be constructed, which satisfy (1). To this end, we consider two non-comoving perfect fluids, which can be interacting or non-interacting, and may or may not admit barotropic equations of state for the fluids. The fluids are chosen to be non-comoving to allow for energy flux and anisotropic pressures in the energy momentum tensor; otherwise the models may be written as a single perfect fluid, which are the models studied in [4, 6]. In a similar vein we consider models with a perfect fluid and scalar field; in contrast to usual work on mixtures of this kind, we allow for the case where the scalar field has a spatial gradient relative to the perfect fluid, which we take to be comoving with the isotropic radiation (test) field. In both cases we take one of the perfect fluids to be ‘comoving’ with the radiation field; that is one of the fluid velocities will be (parallel to) the timelike conformal Killing vector of the spacetime. We then consider some models with non-zero heat flux, but zero anisotropic pressure, which were previously considered in [13], which may be interpreted as ‘quintessence’ models with varying Λ and energy flux.

The case of non-zero rotation has also been considered elsewhere, and simple expanding and rotating spacetimes with plausible matter in which the observers could measure an isotropic CMB have been constructed [10, 11, 9, 14]. This may be considered as a counter-example to numerous claims that the rotation of the universe may be constrained by observations of the CMB alone; such results make additional assumptions of the matter present and the velocity field we follow in the universe.

The upshot of all this is to emphasise that the high isotropy of the CMB when combined with the Copernican principle is simply not enough to draw conclusions about the spatial homogeneity of our universe.

2 Spacetimes admitting an isotropic radiation field

We are interested in spacetimes in which the high isotropy of the CMB is permissible for every observer. In the particular case where we have a model in which all observers on some congruence u^a see an exactly isotropic radiation field, then this velocity field has two important properties:

$$\nabla_{[a} \left(\dot{u}_{b]} - \frac{1}{3} \theta u_{b]} \right) = 0 = \sigma_{ab}. \quad (2)$$

Writing $\dot{u}_a - \frac{1}{3} \theta u_a = \nabla_a Q$ we see that the first condition is equivalent to (1). Spacetimes admitting an isotropic radiation field are conformally stationary, with the velocity fields of the two (conformally related) spacetimes parallel – see the appendix. Now, if we were to assume that these observers measured only dust, then that spacetime must be FLRW – the original EGS theorem [2].

In this paper, for simplicity, we also restrict our attention to the irrotational case. This may be justified by the following considerations. If part of the matter consists of a conserved comoving barotropic perfect fluid other than radiation, or for geodesic motion with any matter source, it follows from (1) that the expansion or the rotation must be zero. For a conserved barotropic perfect fluid, we have $\dot{u}_a = \tilde{\nabla}_a \phi$, and $p' \theta = \dot{\phi}$, where $\phi \equiv - \int dp / (\mu(p) + p)$, and $p' = dp/d\mu$; so, $\eta_{abc} \tilde{\nabla}^b \tilde{\nabla}^c (Q - \phi) = 2(\frac{1}{3} - p') \theta \omega_a = 0$. For geodesic motion, $\eta_{abc} \tilde{\nabla}^b \tilde{\nabla}^c Q = \frac{2}{3} \theta \omega_a = 0$. However, rotating universes which allow an isotropic radiation field have been found and discussed in some detail – see [10, 11, 14, 9].

In this case the metric can then take the form

$$ds^2 = e^{2Q(t, x^\alpha)} \left\{ -dt^2 + H_{\alpha\beta} dx^\alpha dx^\beta \right\} \quad (3)$$

where $H_{\alpha\beta}(x^\gamma)$ can be diagonalised. If $Q = Q(t)$ then the acceleration is zero and we recover the models studied by Coley and MacManus[15]; indeed, even in this case (i.e., the acceleration-free case) it follows that there are physically viable spacetimes that are not FLRW.

In order to find irrotational spacetimes with an isotropic CMB, we can simply compute the Einstein tensor of (3), and equate with the matter we desire. In the appendix, we discuss this computation further.

2.1 The energy momentum tensor of multiple fluids

In general, the energy momentum tensor, T_{ab} for any spacetime may be decomposed with respect to the velocity field u^a in the following covariant manner:

$$T_{ab} = \bar{\mu}u_a u_b + \bar{p}h_{ab} + 2\bar{q}_{(a}u_{b)} + \bar{\pi}_{ab}. \quad (4)$$

This decomposition allows us to make the physical interpretations that $\bar{\mu} = u^a u^b T_{ab}$ is the energy density, $\bar{p} = \frac{1}{3}h^{ab}T_{ab}$ the isotropic pressure, $\bar{q}_a = -h_a^b u^c T_{bc}$ the energy or heat flux and $\bar{\pi}_{ab} = T_{\langle ab \rangle}$ the anisotropic pressure or stress. All these quantities are interpreted by an observer traveling on the u^a congruence.

2.1.1 perfect fluids

Consider the energy-momentum tensor due to two non-comoving perfect fluids;

$$T_{ab} = \mu_1 u_a u_b + p_1 h_{ab} + \mu_2 \tilde{u}_a \tilde{u}_b + p_2 \tilde{h}_{ab}, \quad (5)$$

where μ_i are the energy densities of the fluids in each comoving frame, and the p_i 's are their respective pressures. The velocity field of the second congruence may be written as a Lorentz boost of the first;

$$\tilde{u}^a = \gamma(u^a + v^a), \quad \gamma = \frac{1}{\sqrt{1 - v^a v_a}}, \quad v^a u_a = 0. \quad (6)$$

If we write this as one fluid with respect to the u^a congruence, then T_{ab} has the form of (4), with components

$$\begin{aligned} \bar{\mu} = u^a u^b T_{ab} &= \mu_1 + \mu_2 + \gamma^2 v^2 (\mu_2 + p_2), \\ \bar{p} = \frac{1}{3} h^{ab} T_{ab} &= p_1 + p_2 + \frac{1}{3} \gamma^2 v^2 (\mu_2 + p_2), \\ \bar{q}_a = -h_a^b u^c T_{bc} &= \gamma^2 (\mu_2 + p_2) v_a, \\ \bar{\pi}_{ab} = T_{\langle ab \rangle} &= \gamma^2 (\mu_2 + p_2) v_{\langle a} v_{b \rangle}. \end{aligned} \quad (7)$$

Thus we see that the first fluid will experience an energy flux due to the second fluid passing through their frame (provided $\mu_2 + p_2 \neq 0$).

2.1.2 perfect fluid plus scalar field

In general the energy-momentum tensor of a scalar field ϕ may be written

$$T_{ab}^\phi = \phi_{,a} \phi_{,b} - g_{ab} \left(\frac{1}{2} \phi_{,c} \phi^{,c} + V(\phi) \right), \quad (8)$$

which, when a velocity field is specified, takes the form

$$T_{ab}^\phi = \dot{\phi}^2 u_a u_b + \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi - 2\dot{\phi} u_{(a} \tilde{\nabla}_{b)} \phi - g_{ab} \left(\frac{1}{2} \tilde{\nabla}_c \phi \tilde{\nabla}^c \phi - \frac{1}{2} \dot{\phi}^2 + V(\phi) \right). \quad (9)$$

Thus, if we add to this scalar field a perfect fluid (perfect with respect to this u^a congruence), then we find that the total or mean matter variables become

$$\begin{aligned}\bar{\mu} &= \mu + \mu_\phi = \mu + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi + V(\phi), \\ \bar{p} &= p + p_\phi = p + \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi - V(\phi), \\ \bar{q}_a &= q_a^\phi = -\dot{\phi}\tilde{\nabla}_a\phi, \\ \bar{\pi}_{ab} &= \pi_{ab}^\phi = \tilde{\nabla}_{\langle a}\phi\tilde{\nabla}_{b\rangle}\phi.\end{aligned}\tag{10}$$

Note that if $\tilde{\nabla}_a\phi = 0$ then formally the total fluid takes the form of a single perfect fluid.

We may demand that ϕ satisfies the Klein-Gordon equation, which may be derived from the energy conservation equation for the scalar field, $\nabla^a T_{ab}^\phi = 0$,

$$\frac{\partial V(\phi)}{\partial \phi} = \nabla_a \nabla^a \phi = \dot{u}^a \tilde{\nabla}_a \phi + \tilde{\nabla}^a \tilde{\nabla}_a \phi - \ddot{\phi} - \theta \dot{\phi}.\tag{11}$$

However, if there is an interaction between the scalar field and some other matter, then this equation may not hold; for example, we could have a scalar field decaying into physical matter in which case the energy of the scalar field will not be conserved independently.

2.2 Scalar Tensor Theories of Gravity

Scalar-tensor theories, in which a long-range scalar field combined with a tensor field mediate the gravitational interaction, are standard alternatives to general relativity. The original motivation for these theories was to incorporate a varying gravitational constant into GR to account for alleged discrepancies between observations and weak-field predictions of GR. A special case of the scalar tensor theories, known as the Brans-Dicke theory of gravity (BDT [16]) (with a constant ω_0 parameter), was the original of these theories. Scalar-tensor theories occur as the low-energy limit in supergravity theories from string theory [17] and other higher-dimensional gravity theories [18].

Recently the recovery of the EGS theorem in scalar tensor theories was given [19]; geodesic observers in a scalar tensor theory of gravity observing isotropic radiation must be in a FLRW universe. We mention here that inhomogeneous spacetimes are possible however if the geodesic assumption is dropped. The field equations, obtained by varying the BD action with respect to the metric and the field ϕ , are

$$G_{ab} = \frac{8\pi}{\phi} T_{ab} + \frac{\omega}{\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right) + \phi^{-1} (\nabla_a \nabla_b \phi - g_{ab} \nabla_c \nabla^c \phi) - \frac{1}{2} g_{ab} U(\phi).\tag{12}$$

$$(3 + 2\omega) \nabla_a \nabla^a \phi = 8\pi T - \nabla_a \omega \nabla^a \phi + \frac{dU}{d\phi}\tag{13}$$

where the energy-momentum tensor of the matter, T^{ab} , may take any of the usual desired forms. In the scalar-tensor gravity theories the principle of equivalence is guaranteed by requiring that all matter fields are minimally coupled to the metric g_{ab} . Thus energy-momentum is conserved:

$$\nabla^a T_{ab} = 0.\tag{14}$$

It is known that scalar-tensor theories can be rewritten in the conformally related ‘Einstein’ frame [20], so that the models are formally equivalent to GR coupled to a scalar field. Therefore, scalar-tensor theories may be incorporated here as a special case of the scalar field in GR – see Sec. 2.1.2.

3 Some solutions

Rather than provide an exhaustive study of multi-fluid solutions, we will present some example of how such solutions may be derived. This is in keeping with our aim of illustrating the existence of ‘realistic’ inhomogeneous cosmological solutions which satisfy the Copernican principle.

3.1 Two perfect fluids

For simplicity, we restrict ourselves to the case where $Q(t, x^\alpha) = Q(t, r)$ in comoving polar coordinates and we also restrict ourselves to the spherically symmetric case, $H_{rr} = H_{\theta\theta} = H_{\phi\phi} = \exp 2B(r, \theta, \phi)$. In this case the energy flux relative to the comoving u^a frame is

$$\bar{q}_r = 2e^{-Q} (Q_{,tr} - Q_{,t}Q_{,r}) = \gamma^2(\mu_2 + p_2)v_r, \quad \bar{q}_\theta = \bar{q}_\phi = 0 \Rightarrow v_\theta = v_\phi = 0. \quad (15)$$

Because v^a has one non-vanishing component, this implies from (7) that $\bar{\pi}_{ab}$ must be diagonal. Calculating the components of π_{ab} , we find that B can only be a function of r alone:

$$\begin{aligned} \bar{\pi}_{rr} &= \frac{2}{3} \left[-(B'' + 2Q'') + (B' + 2Q')^2 - 2Q'^2 + \frac{1}{r}(B' + 2Q') \right] \\ &= \frac{2}{3} \gamma^2 (\mu_2 + p_2) v_r^2. \end{aligned} \quad (16)$$

Hence we find

$$v_r = \frac{3}{2} \frac{\bar{\pi}_{rr}}{\bar{q}_r}. \quad (17)$$

We may also calculate the mean energy density and pressure;

$$\begin{aligned} \bar{\mu} &= 3e^{-2Q} Q_{,t}^2 - e^{-2(Q+B)} \left[2(B'' + Q'') + (B' + Q')^2 + \frac{4}{r}(B' + Q') \right], \\ &= \mu_1 + \mu_2 + \gamma^2 v^2 (\mu_2 + p_2), \\ \bar{p} &= 3e^{2Q} \left[Q_{,t}^2 + 2Q_{,tt} \right] - e^{-2(Q+B)} \left[2(B'' + Q'') + B'^2 + 5Q'^2 + 4Q'B' + \frac{4}{r}(B' + 2Q') \right] \\ &= p_1 + p_2 + \frac{1}{3} \gamma^2 v^2 (\mu_2 + p_2). \end{aligned} \quad (18)$$

As yet we have only determined v (and γ), and we have three equations relating four functions, μ_i, p_i (Eqs. (18), and the remaining freedom from $\bar{\pi}_{rr}$ and q_r). In principle we have the freedom to specify one more equation relating the four free functions. The most obvious restrictions are barotropic equations of state for the two fluids, $p_i = p_i(\mu_i)$, or separate energy conservation for the two fluids, $\nabla^a T_{ab}^i = 0$. However, if Q and B are specified then only one of these types of conditions may be used in general. We can use the freedom in the two metric functions to allow us to use both conditions if we choose. For simplicity, we consider the case of the two fluids obeying linear equations of state, $p_i = w_i \mu_i$.

3.1.1 Example: $p_i = w_i \mu_i$, with $B = 0$

We have four free functions, Q, μ_i , and v_r together with four equations; specifying two equations of state is then sufficient to close the system. We may solve Eqs. (15) and (16) for μ_2 and v_r as functions of Q

(and derivatives of). Substituting these into (18) we get two equations for μ_1 ; requiring equality leads to a horrendous equation for Q ;

$$\begin{aligned}
0 = & ((2r^2 + 2w_1r^2)Q_{rr}^2 + ((-w_1r^2 - r^2 + 3w_2w_1r^2 + 3w_2r^2)Q_r^2 + (6w_2w_1r + 2w_1r \\
& + 4w_2r)Q_r + (-2w_2r^2 - 2r^2)Q_{tt} + (-w_2r^2 - 3w_2w_1r^2 - r^2 - 3w_1r^2)Q_t^2)Q_{rr} \\
& + (-3w_2r^2 - w_1r^2 - 3w_2w_1r^2 - r^2)Q_r^4 + (-7w_2r - 5w_1r - 3r - 9w_2w_1r)Q_r^3 \\
& + ((2w_2r^2 + 2r^2)Q_{tt} + (3w_2r^2 + r^2 + w_1r^2 + 3w_2w_1r^2)Q_t^2 - 4w_2 - 4w_1 - 2 - 6w_2w_1)Q_r^2 \\
& + ((2w_2r + 2r)Q_{tt} + (3w_1r + w_2r + 3w_2w_1r + r)Q_t^2 + (-4w_2r^2Q_{tr} + 4w_1r^2Q_{tr})Q_t)Q_r \\
& + 2w_2r^2Q_{tr}^2 - 2w_1r^2Q_{tr}^2)/rw_1(-Q_r - rQ_r^2 + rQ_{rr})(1 + w_2). \tag{19}
\end{aligned}$$

A solution of the form $Q = a \ln t + b \ln r$ exists, provided we choose

$$a = -2 \frac{1 + w_2^2 + 2w_2}{-8w_2w_1 - 2w_2 - 3w_1 + 3w_2^2 - 2w_1^2 + 3w_2^2w_1 - 6w_2w_1^2 - 1}, \tag{20}$$

$$b = -2 \frac{3w_2w_1 + 2w_2 + w_1}{1 + 3w_2 + w_1 + 3w_2w_1}. \tag{21}$$

The only other physical constraint is that $v^2 < 1$, for all t and r . In the case where the first fluid is dust, $w_1 = 0$, this requires that $-\frac{1}{3} < w_2 < -\frac{1}{6}$. This also ensures that the fluids become comoving at late times.

Thus we have demonstrated the existence of inhomogeneous two barotropic fluid solutions of the field equations which allow the existence of isotropic radiation. There are clearly much more general solutions than we have presented here, our solution being a very special case.

3.2 Perfect fluid plus scalar field revisited.

As before, we restrict ourselves to the spherically symmetric case where $Q(t, x^\alpha) = Q(t, r)$ and $H_{rr} = H_{\theta\theta} = H_{\phi\phi} = \exp(2B(r, \theta, \phi))$. In this case the energy flux relative to the comoving u^a frame is

$$\bar{q}_r = 2e^{-Q}(Q_{,tr} - Q_t Q_r) = -\dot{\phi} \tilde{\nabla}_r \phi, \quad \bar{q}_\theta = \bar{q}_\phi = 0 \Rightarrow \phi(t, r). \tag{22}$$

Because ϕ is only a function of time and the spatial coordinate r , this implies that $\bar{\pi}_{ab}$ must be diagonal. Calculating the components of $\bar{\pi}_{ab}$, we find that B can only be a function of r alone. We find that

$$\frac{2}{3}(\tilde{\nabla}_r \phi)^2 = \bar{\pi}_{rr} = \frac{2}{3}[-(B'' + 2Q'') + 2Q'^2 + B'^2 + 4B'Q' + \frac{1}{r}(2Q' + B')]. \tag{23}$$

Hence we find

$$\dot{\phi}^2 = \frac{2}{3} \frac{\bar{q}_r^2}{\bar{\pi}_{rr}}.$$

We may also calculate the mean energy density and pressure;

$$\begin{aligned}
\bar{\mu} &= 3e^{-2Q}\ddot{Q}^2 - e^{-2(Q+B)}[2(B'' + Q'') + (B' + Q')^2 + \frac{4}{r}(B' + Q')] \\
&= \mu + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\tilde{\nabla}_r \phi \tilde{\nabla}^r \phi + V(\phi), \tag{24}
\end{aligned}$$

$$\begin{aligned}
\bar{p} &= -\frac{1}{3}[3e^{-2Q}[\dot{Q}^2 + 2\ddot{Q}] - e^{-2(Q+B)}[4Q'' + 2B'' + 4B'Q' + 5Q'^2 + B'^2 + \frac{4}{r}(B' + 2Q')]] \\
&= p + \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\tilde{\nabla}_r\phi\tilde{\nabla}^r\phi - V(\phi),
\end{aligned} \tag{25}$$

We may impose the restriction of a barotropic equation of state for the perfect fluid $p(\mu)$ and an energy conservation law for the fluid $\nabla^a T_{ab} = 0$. We choose the equation of state $p = w\mu$. The energy conservation equations for the perfect fluid read

$$\mu_{,t} + 3Q_{,t}(1+w)\mu = 0 \tag{26}$$

$$w\mu' + Q'(1+w)\mu = 0, \tag{27}$$

which when integrated imply that Q must have the form

$$Q(t, r) = \alpha(t) + \beta(r). \tag{28}$$

It follows that a specific form for the comoving energy will be

$$\mu = \exp[-(1+w)(3\alpha + \beta/w)]. \tag{29}$$

To simplify things we will assume that $B = 0$, so that the metric now becomes

$$g_{ab} = e^{2\alpha(t)+2\beta(r)}\eta_{ab}; \tag{30}$$

which implies that $\theta = 3\alpha_{,t}e^{-Q}$, $\dot{u}_r = \beta'$, and

$$\bar{\pi}_{rr} = \frac{2}{3}\left[-2\beta'' + 2\beta'^2 + \frac{2}{r}\beta'\right] \tag{31}$$

$$\bar{q}_r = -2e^{-Q}\alpha_{,t}\beta'. \tag{32}$$

Hence we find that

$$\dot{\phi}^2 = \frac{2}{3}\frac{\bar{q}_r^2}{\bar{\pi}_{rr}} = \frac{2}{9}\theta^2\beta'^2\frac{1}{[\beta'^2 + \frac{\beta'}{r} - \beta'']}. \tag{33}$$

The scalar field wave equation is

$$-\ddot{\phi} + \phi''e^{-2B} - 2\dot{Q}\dot{\phi} + (2Q' + B')\phi'e^{-2B} + \frac{2\phi'}{r}e^{-2B} = \frac{dV}{d\phi}e^{2Q}. \tag{34}$$

We will assume a solution of the form

$$\phi = \Phi(t) + \Psi(r), \tag{35}$$

which allows the derivation of the following two equations from the scalar field equation when $V = 0$:

$$\ddot{\Phi} + 2\dot{\alpha}\dot{\Phi} = C \tag{36}$$

$$\Psi'' + 2(\beta' + \frac{1}{r})\Psi' = C, \tag{37}$$

where C is a constant. We can rewrite the equations for heat conduction and anisotropic pressure as

$$\dot{\Phi}\Psi' = 2e^{-Q}\dot{\alpha}\beta' \tag{38}$$

$$\Psi'^2 = -2\beta'' + 2\beta'^2 + \frac{2}{r}\beta' \quad (39)$$

Hence we compute the following differential equations for α and β :

$$\ddot{\alpha} + \dot{\alpha}^2 = Ae^\alpha \quad (40)$$

$$\beta'' + \beta'^2 + 2\frac{\beta'}{r} = Be^\beta \quad (41)$$

These equations have non-trivial solutions implying a non-FLRW cosmology (the FLRW limit is recovered when $\beta = 0$). Hence we have shown that spacetimes with a barotropic perfect fluid and a *non-comoving* scalar field exist which allow an isotropic radiation field for all observers, which are non-FLRW. Clearly there are a huge number of solutions meeting this criteria; we have demonstrated existence in this simplest of cases. These new solutions could play an important role in cosmology, for example as a new generalisation of quintessence.

4 Inhomogeneous quintessential cosmologies

Simple multifluid models can be constructed by introducing, together with a barotropic fluid, a varying $\Lambda(t)$ term as the second fluid:

$$\rho = \mu + \Lambda, \quad p = (\gamma - 1)\mu - \Lambda \quad (42)$$

where ρ , p are “total” energy density and pressure obtained from the field equations. The varying Λ term can be interpreted as the asymptotic state of a scalar field associated with a quintessence dominated scenario, coexisting with a material fluid described by μ . We show in this section that such an interpretation is compatible with the asymptotic properties of a class of simple models that allow an isotropic radiation field.

The simplest models satisfying the EGS criterion and compatible with the decomposition (42) are characterized by the conformally FLRW metric (a particular case of (3)) given by

$$ds^2 = \frac{-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)}{\Phi^2}, \quad \Phi \equiv a(t) + b(t)r^2 \quad (43)$$

whose source is the imperfect fluid

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a} u_{b)} \quad (44)$$

where $u^a = \Phi \delta^a_t$ and $q_a = q_r \delta^r_a$, while ρ and p must comply with (42). Applying the field equations for (43) and (44) leads to

$$8\pi \gamma \mu = 2 \left(a_{,tt} + 2b + b_{,tt} r^2 \right) (a + br^2) \quad (45)$$

$$8\pi q_r = -4b_{,t}r, \quad 8\pi q = 8\pi |g^{ab} q_a q_b|^{1/2} = 4|b_{,t}|r(a + br^2) \quad (46)$$

$$8\pi \gamma \Lambda = L_4(t) r^4 + 2L_2(t) r^2 + L_0(t) \quad (47)$$

where

$$L_4(t) \equiv 3b_{,t}^2 \gamma - 2b_{,tt}b, \quad L_2(t) \equiv 3\gamma a_{,t}b_{,t} - 2b^2 - a_{,tt}b - b_{,tt}a, \quad L_0(t) \equiv 3\gamma (a_{,t}^2 + 4ab) - 2a(2b + a_{,tt}) \quad (48)$$

If we demand that Λ be only a time-dependent function, we obtain

$$L_2(t) = 0, \quad L_4(t) = 0, \quad 8\pi \Lambda = L_0(t)/\gamma \quad (49)$$

which yields differential equations that determine a , b for a given $\gamma(t)$, and the definition of $\Lambda(t)$.

Since we are interested in an asymptotic regime that assumes a slowly varying $\gamma(t)$, we shall consider a constant γ . The general solution of the system (49) in this case is

$$b = b_0 t^{-2\nu}, \quad a = a_1 t^{-2\nu} + a_2 t^{-3\gamma\nu} - \frac{b_0}{3} t^{6(\gamma-1)\nu}, \quad \gamma \neq 2/3, \quad \nu \equiv 1/(3\gamma - 2) \quad (50)$$

where b_0 , a_1 , a_2 are arbitrary integration constants. Since we are assuming μ characterizes a material fluid (baryons plus photons and possibly CDM), we have that $1 \leq \gamma \leq 2$ and hence ν in (50) is always positive. Inserting (50) into (43), (45), (46), and (49) leads to

$$\frac{1}{\Phi} = \frac{1}{a + br^2} = \frac{t^{3\gamma\nu}}{(-b_0/3)t^3 + (a_1 + b_0 r^2)t + a_2} \quad (51)$$

$$8\pi \mu = \frac{12 [(2\gamma - 5/3)b_0 t^3 + (a_1 + b_0 r^2)t + (3\gamma - 1)a_2] [(-b_0/3)t^3 + (a_1 + b_0 r^2)t + a_2]}{\nu^2 t^{4(3\gamma-1)\nu}} \quad (52)$$

$$8\pi q_r = \frac{2}{\nu} r t^{-3\gamma\nu}, \quad 8\pi q = \frac{8|b_0| [(-b_0/3)t^3 + (a_1 + b_0 r^2)t + a_2] \nu r}{t^{6\gamma\nu}} \quad (53)$$

$$8\pi \Lambda = \frac{(-8/3)b_0^2 t^6 + 12a_1 b_0 t^4 + 16a_2 K t^3 + 3a_2^2}{t^{4(3\gamma-1)\nu}} \quad (54)$$

Because of the apparent (and coordinate dependent) resemblance of (43) to a spatially flat FLRW, it is tempting to assume that these two metrics have common geometric features. For example, it is evident from (52), (53) and (54) that μ , Λ and q diverge as $t \rightarrow 0$ for $\nu > 0$ and $1 \leq \gamma \leq 2$, hence we can identify $t = 0$ as the locus of a “big bang” singularity, analogous to the FLRW big bang. In spatially flat FLRW spacetimes it is always possible to assume that the coordinate range is given by $0 < t < \infty$ and $0 \leq r < \infty$, so that $t \rightarrow \infty$ and $r \rightarrow \infty$ mark asymptotic future infinities in the timelike, null and spacelike directions. However, for the models under consideration the coordinate domain is necessarily restricted by the extra condition that the conformal factor $1/\Phi$ be a bounded function. Also, proper time along the worldlines of comoving observers is $\tau = \int dt/\Phi$ evaluated for fixed (r, θ, ϕ) , and so a sufficient condition for having $\tau \rightarrow \infty$ occurs if the conformal factor $1/\Phi$ diverges, even if it does so for finite values of the coordinates t, r . From (51), this occurs for all a_1, a_2, b_0, γ , since the equation $(-b_0/3)t^3 + (a_1 + b_0 r^2)t + a_2 = 0$ always has real roots in the coordinate domain $t > 0, r \geq 0$, defining the hypersurface

$$B = [t, r(t), \theta, \phi], \quad r(t) = \left[\frac{(1/3)b_0 t^3 - a_1 t - a_2}{b_0 t} \right]^{1/2} \quad (55)$$

which can be represented as a parametric curve in the t, r plane. If one or both of a_1, a_2 is zero, the boundary B persists, though its parametrization in the t, r plane is simpler than (55). The only exception is if $b_0 = 0$, whence the solutions trivially reduce to FLRW. Therefore, this feature is inherent to the models characterized by (42), (43) and (44).

The fact that Φ^{-1} and $Y = r\Phi^{-1}$ diverge at (55) means that B marks a spacetime boundary beyond which the spacetime manifold cannot be extended. An asymptotic past/future is then defined as the coordinate values marked by B which are reached by causal curves, either comoving observers ($r = \text{const.}$) or

radial null geodesics: ($v = t + r$, $w = t - r$). From (51), it is straightforward to prove that $\tau \rightarrow \infty$ holds as B is reached by future and past directed worldlines of comoving observers. Also the affine parameter of radial null geodesics diverge at spacetime points marked by B . The coordinate domain of definition is then restricted by $|\Phi| > 0$ and depends on the signs of the constants a_1 , a_2 , b_0 , specifying the form of B in the plane t, r . From the various numerical values for these constants, we eliminate all those cases in which the evolution of the comoving observers occurs between two branches of B . The remaining cases display the two types of evolution classified below:

Case (i). If a_1 , a_2 , b_0 are negative, B lies in the infinite past of all observers evolving towards their infinite future as $t \rightarrow \infty$, and we have a null infinity analogous to that of a FLRW cosmology (the infinite past is then marked by B). Using null coordinates the asymptotic limit along outgoing radial null geodesics $v \rightarrow \infty$ is given by

$$8\pi\mu \rightarrow \frac{16b_0^2(3\gamma-1)}{9\nu} v^{2(3\gamma-4)\nu}, \quad 8\pi q \rightarrow \frac{16b_0^2\nu}{3} v^{2(3\gamma-4)\nu}, \quad \Lambda \rightarrow -\frac{8b_0^2}{3} v^{2(3\gamma-4)\nu} \quad (56)$$

so that a regular null infinity requires $\gamma > 4/3$ (otherwise, the affine parameter has a finite limit as $t \rightarrow \infty$ and this locus marks a null singularity). For a heat conducting shear-free fluid the weak energy condition requires: $\rho + p = \gamma\mu > 2q$ [21], a relation that is satisfied by the asymptotic forms (56) only for $2/3 < \gamma < 1$. Hence, this case is unphysical.

Case (ii) If a_1 , a_2 , b_0 are positive, then all worldlines of comoving observers start their evolution at $t = 0$ (big bang) and evolve towards their infinite future at B . From (52), (53) and (54), we have

$$\frac{q}{\mu} = \frac{2\nu^3|b_0|t^2r}{(2\gamma-5/3)b_0t^3 + (a_1+b_0r^2)t + (3\gamma-1)a_2} \quad (57)$$

$$\frac{\Lambda}{\mu} = \frac{\nu^2 [(-(8/3)b_0t^6 + 12a_1t^4 + 16a_2t^3)b_0 + 3a_2^2]}{12[(2\gamma-5/3)b_0t^3 + (a_1+b_0r^2)t + (3\gamma-1)a_2][(-b_0/3)t^3 + (a_1+b_0r^2)t + a_2]} \quad (58)$$

so that near $t = 0$ we obtain

$$\frac{q}{\mu} \rightarrow \frac{2\nu^3r}{(2\gamma-5/3)t}, \quad \frac{\Lambda}{\mu} \rightarrow \frac{\nu^2}{4(3\gamma-1)} \quad (59)$$

while at the boundary B we have

$$\mu \rightarrow 0, \quad q \rightarrow 0, \quad \rho \rightarrow \Lambda, \quad p \rightarrow -\Lambda, \quad \Lambda \rightarrow \Lambda_B = \Lambda(t_B) \quad (60)$$

$$\left[\frac{q}{\mu}\right]_B = \frac{3\nu^4\sqrt{|b_0|}t_B^2r_B}{2b_0t_B^3 + a_2}, \quad \Lambda_B = \frac{4b_0t_B^2[(a_1-2b_0r_B^2)t_B + 2a_2]}{t_B^{4(3\gamma-1)\nu}} \quad (61)$$

where t_B , r_B are related by (55). The limits (59) indicate (for $1 \leq \gamma \leq 2$) that the models are matter dominated at the big bang ($0 < \Lambda \ll \mu$), evolving towards a Λ dominated future at B (represented by (61). Notice that the asymptotic future state at B can be de Sitter (or anti de Sitter) and is not an asymptotically homogeneous state since Λ_B depends on position (for each observer a different constant value). For $r_B = 0$ (in (55)) we have that $\Lambda_B > 0$. As r_B grows along B , $\Lambda_B < 0$, which implies that the total energy density $\rho = \mu + \Lambda$ becomes negative asymptotically. Hence, for the physical reasons, we exclude coordinate values $r > \bar{r}_B$, where \bar{r}_B satisfies $\Lambda_B(\bar{r}_B) = 0$.

The behavior of q is compatible with the energy conditions, since $q \approx \mu r/t \ll \mu$ holds all along the evolution, near the big bang and near B . It is still necessary to find an adequate physical interpretation for

this term, whether as a heat flux or as a kinetic term associated with a velocity field or the dipole of a kinetic theory distribution [22]. However, since we are interested mainly in the asymptotic stage near B , as long as r is sufficiently small we will have $q \ll \mu$ and could consider q as a residual term.

Another feature of the models, absent in FLRW spacetimes, is the fact that μ and q diverge as $r \rightarrow \infty$ along hypersurfaces of constant t that do not intersect B , marking a point singularity at spacelike infinity. This feature is also present for perfect fluid sources of (43), see [23]. Note that models similar to those examined here were considered recently [13]; however, the existence of the boundary (55) was not considered in the asymptotic study of those models.

The models discussed in this section illustrate how even simple inhomogeneous spacetimes have a much richer geometrical structure that heavily constrains their physical applicability. Pending a reasonable physical interpretation for q_a and provided we exclude sufficiently large values of r , these solutions are inhomogeneous models that comply with the EGS criterion and describe a Λ dominated scenario usually associated with the “quintessence” field [24, 25, 26].

5 Discussion

In this paper we have proven the existence and examined the physical viability of a number of spacetimes which have been constructed to allow an isotropic radiation field. Since these inhomogeneous spacetimes satisfy the Copernican principle (as far as the CMB is concerned), the question of finding methods of testing the cosmological principle, and thus observationally testing whether the universe is in fact an FLRW model, arises.

More precisely, it has been shown here and elsewhere [4, 6] that inhomogeneous universe models with non-geodesic observers obey the EGS criterion. That is, inhomogeneous spacetimes have been found which allow every observer to see an isotropic CMB. It has also been shown that a significant subset of these models are consistent with other observational constraints, and hence these models are consistent with observations even when the Copernican principle is taken into account – and yet the models are not spatially homogeneous [6]. A potential problem with these particular models is that the non-geodesic motion of the observers prohibits a barotropic equation of state for perfect fluid matter. However, we have shown here that more general and physically viable cosmological models (with realistic matter) allow an isotropic radiation field. In particular, irrotational multi-fluid spacetimes have been constructed which satisfy (1). These cosmologies include two non-comoving perfect fluids, which can be interacting or non-interacting, and may or may not admit barotropic equations of state for the fluids. The fluids are chosen to be non-comoving to allow for energy flux and anisotropic pressures in the energy momentum tensor (otherwise the models may be written as a single perfect fluid and correspond to the models studied in [4, 6]). Even in the acceleration-free case there are examples of spacetimes that are not FLRW [15]. Similarly, models with a perfect fluid and scalar field can be constructed in which the scalar field can have a spatial gradient relative to the perfect fluid, which is taken to be comoving with the isotropic radiation field. As a particular example, a class of shear-free spherically symmetric, inhomogeneous (quintessential) cosmologies whose source is a heat conducting fluid and a scalar field were considered in detail.

One of our key assumptions has been zero rotation. It has been shown that rotating spacetimes which allow an isotropic radiation field may also be constructed [10, 11, 14, 9].

Other recent work also supports these conclusions. In the fundamental EGS theorem [2], and here, it is assumed that all fundamental observers measure the CMB temperature to be exactly isotropic during a time interval I (defined by $t_E \leq t \leq t_0$, where t_E is the time of last scattering and t_0 is the time of observation). Under this assumption the theorem then asserts that the universe is *exactly* an FLRW model during this time

interval. However, the EGS theorem cannot be used to conclude that the physical universe is close to an FLRW model since the CMB temperature can only be observed at one instant of time on a cosmological scale. Hence it is of interest to ask what restrictions, if any, can be placed on the anisotropy in the rate of expansion, assuming that all fundamental observers measure the CMB temperature to be exactly isotropic at *some instant of time* t_0 only. On the basis of continuity, it can then be argued that all fundamental observers will measure the CMB temperature to be almost isotropic in some time interval of time of length δ centered on t_0 . This time interval could, however, be much shorter than the time interval I . However, in [12] it was shown that *for a given time* t_0 , there is a class of locally rotationally symmetric non-tilted dust Bianchi type VIII spatially homogeneous cosmological models such that at t_0 the CMB temperature is measured to be isotropic by all fundamental observers, even though the overall expansion of the universe is highly anisotropic at t_0 .

In addition, the EGS theorem is of course not directly applicable to the real universe since the CMB temperature is not exactly isotropic. This result has consequently been generalized by [27] to the “almost EGS theorem”, which states that if all fundamental observers measure the CMB temperature to be almost isotropic during some time interval in an expanding universe, then the universe is described by an almost FLRW model during this time interval. The dimensionless *shear parameter* and the *Weyl parameter* were introduced in [28]. Since the Weyl curvature tensor is related to time derivatives of the shear tensor, restricting the shear parameter to be small does *not* guarantee that the Weyl parameter is small. Therefore a necessary condition for the universe to be close to an FLRW model is that both of the shear and Weyl parameters must be small. In the almost-EGS theorems the dimensionless time and spatial derivatives of the multipoles are assumed to be bounded by the multipoles themselves [27]. If this assumption is not satisfied, then the CMB temperature observations *do not impose upper bounds on the shear and Weyl parameters*, and hence do not establish that the universe is close to FLRW. In [29] a class of spatially homogeneous non-tilted Bianchi type VII₀ dust models in which the CMB is treated as a test field or a non-interacting radiation fluid was studied. To obtain the present CMB temperature pattern, the photon energies were integrated numerically along the null geodesics that connect points of emission on the surface of last scattering with the event of observation at the present time. Wainwright et al. [30] then showed that the shear parameter tends to zero but the Weyl parameter does not tend to zero at late times in these models. In other words, *although the models isotropize as regards the shear, the Weyl curvature remains dynamically significant*. A variety of numerical simulations to calculate anisotropy patterns of the CMB temperature in Bianchi VII₀ models were explicitly performed [29] to demonstrate that there exists cosmological models that are not close to any FLRW model even though the temperature of the CMB is almost isotropic in the sense that the observational bounds on the quadrupole and octupole are satisfied.

It is clear, then, that the Copernican principle when combined with our observations of the CMB does not imply the cosmological principle: that the universe is homogeneous and isotropic. Recent work on this suggests that the assumptions in the EGS (dust observers) and almost-EGS (small gradients of CMB multipoles and dust observers) theorems are crucial to the conclusions; weakening of any of these assumptions appears to negate the theorems almost entirely. It is therefore important to ask what observations we need to test the cosmological principle?

Assuming for the moment that the assumptions of the almost EGS theorem actually hold in our universe, then one method by which the cosmological principle may be tested is as follows: if we can observe the CMB as seen by some other observers, then we can immediately confirm or reject the cosmological principle. That is, if we find the CMB is as isotropic around these other observers as we see it around us, we may conclude that we live in a homogeneous universe. There is, in fact, a physical method by which we *can* observe the CMB as seen by other observers. It consists of light from the CMB being scattered by hot gas in galaxy

clusters in such a way as to allow us to observe the anisotropy of the CMB as seen by that particular galaxy. This is known as the Sunyaev-Zel'dovich (SZ) effect [31], and has been suggested by a number of authors as a possible means to test the Copernican and cosmological principles [32].

This will only work of course if the initial assumptions of the EGS theorem apply in our actual Universe. The original EGS theorem relies on the observers in the universe being well described by a dust fluid – i.e., they are geodesic. Indeed, in the almost EGS theorem, it was necessary to demand dust observers to first order – i.e., more general matter was only allowed at second order.

Any possible attempt to verify the cosmological principle by using methods such as the SZ effect above, or any other method which relies on making observations of the CMB from other locations, will fail. If we were living in a universe found here or in [4], for example, we would see exactly the same effect: all observations of the CMB around other observers would be as isotropic as the standard homogeneous FLRW models. It follows, therefore, that *the high isotropy of the CMB can never be used, on its own, to show our universe is nearly homogeneous.*

On the other hand, we *can* test the cosmological principle using the SZ effect and methods like it *provided* we can show definitively that our universe is made of a dust-like fluid and that we travel on geodesics. Recent observations suggesting quintessential matter making up a significant part of the energy density of the universe throws this standard assumption into question: there is no *a priori* reason why this matter – whatever it turns out to be – should be homogeneous (many dark matter theories allow equations of state which are not dust also).

The geodesic assumption can be tested to some degree by local observations. If we look closely at the recession velocity of galaxies close to us, then we can detect a dipole moment in the relative velocities of these galaxies. This deviation from the linear Hubble law is usually attributed to our local random motion with respect to the local expansion rate of the universe, caused by our gravitational infall into the Great Attractor. Acceleration will also leave its mark on the linear Hubble law also as a dipole distribution in the direction of the CMB dipole, but as a dipole which grows linearly with distance [4]. With sufficiently accurate knowledge of the local distribution of galaxies, the two effects can be disentangled as our bulk gravitational motion affects the Hubble law irrespective of distance. Current knowledge of our local group motion will only provide relatively weak constraints on acceleration (roughly $\dot{u}/H_0 \sim 0.1$; compare this with $\sigma/H_0^2 \sim 10^{-5}$ from almost-EGS CMB observations [33]).

If we assume, for arguments sake, that the results of such a study reveal that the local dipole is entirely accounted for by our peculiar velocity, then what can we say about the spatial homogeneity of our universe? If the acceleration around our location is small, then we may assert the Copernican principle and ascertain that all observers in the universe follow geodesics; therefore we may apply the EGS theorem (assuming the SZ measurements measure small enough anisotropies of the CMB around other observers) and deduce that the cosmological principle is a valid assumption. This would yield tremendous support for our faith in the standard model.

Of course, acceleration is just one possible inhomogeneity which causes problems: others are rotation, Weyl curvature, and anisotropy of the energy momentum tensor. These all leave their mark to varying degrees to the anisotropy of the magnitude-redshift and number-count-redshift relations, but at higher order in redshift (if expanded in a power series in redshift) than acceleration, so are much harder to detect.

What should be clear from this argument is that the cosmological principle is simply untestable. Even if SZ measurements reveal the Copernican principle to be true, accurate determination of anisotropies of the magnitude-redshift relation at high redshift are out of the question for the foreseeable future.

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A. Appendix. The energy momentum tensor of conformally related spacetimes

A conformal transformation is an angle preserving transformation that changes lengths and volumes. The importance of these types of transformations lies in the fact that, under a conformal transformation, *the causal structure of the spacetime is preserved*. The Weyl tensor, $C_{abc}{}^d$, is invariant, so that a conformal transformation will introduce no tidal forces or gravitational waves; that is, a conformal transformation will only introduce ‘non-gravitational’ forces and matter into the new spacetime (by changing R_{ab} and thus the matter tensor T_{ab} via Einstein’s equations).

We will discuss conformal transformations and their 1+3 splitting here as it is a useful tool for constructing new spacetimes from old, especially if one is after spacetimes with an isotropic radiation field. The conformal transformation is defined by

$$g_{ab} = e^{2Q} \hat{g}_{ab}, \quad u^a = e^{-Q} \hat{u}^a, \quad u_a = e^Q \hat{u}_a; \quad (62)$$

where $Q > 0$ is an arbitrary function, u^a is a velocity vector with respect to g_{ab} : $g_{ab}u^a u^b = u_a u^a = -1$; and \hat{u}^a is the conformally related (parallel) velocity vector, and is normalised with respect to \hat{g}_{ab} : $\hat{g}_{ab}\hat{u}^a \hat{u}^b = -1$. The covariant derivative of any one-form field v_a transforms as

$$\nabla_a v_b = \hat{\nabla}_a v_b - 2Q_{(a} v_{b)} + g_{ab} Q^c v_c, \quad (63)$$

where $Q_a \equiv Q_{,a} = \tilde{\nabla}_a Q - \dot{Q}u_a$. The expansion ($\theta = \text{div } u$), acceleration ($\dot{u}^a = u^b \nabla_b u^a$), rotation ($\omega_a = -\frac{1}{2} \text{curl } u_a$), and shear ($\sigma_{ab} = \tilde{\nabla}_{(a} u_{b)}$) of the two velocity congruences are related by:

$$\begin{aligned} \hat{\theta} &= e^Q (\theta - 3\dot{Q}) \\ \hat{u}_a &= \dot{u}_a - \tilde{\nabla}_a Q \\ \hat{\omega}_a &= \omega_a \\ \hat{\sigma}_{ab} &= e^{-Q} \sigma_{ab}. \end{aligned} \quad (64)$$

The equation for the acceleration corrects equation (6.14) of [34]. These show that a conformal transformation may induce acceleration and expansion into the new spacetime, but not shear or rotation: in particular, a conformally flat model must have vanishing shear and rotation [4, 6]. With respect to g_{ab} , a dot denotes time differentiation along the fluid flow, $\dot{F}_{ab} = u^c \nabla_c F_{ab}$, and $\tilde{\nabla}_a$ is the spatial derivative projected orthogonal to the flow lines, $\tilde{\nabla}_c F_{ab} = h^d{}_c h^e{}_a h^f{}_b \nabla_d F_{ef}$, where $h_{ab} = g_{ab} + u_a u_b$ is the usual projection tensor.

The Einstein tensor transforms as

$$G_{ab} = \hat{G}_{ab} - 2\nabla_a Q_b - 2Q_a Q_b + g_{ab} [2\nabla_c Q^c - Q^2], \quad (65)$$

where \hat{G}_{ab} is the Einstein tensor of \hat{g}_{ab} , and $Q^2 = Q_a Q^a$. For clarity, we decompose derivatives of Q into time and space derivatives:

$$\begin{aligned} Q_a &= \tilde{\nabla}_a Q - \dot{Q}u_a, \\ \nabla_b Q_a &= u_a u_b (\ddot{Q} - \dot{u}^c \tilde{\nabla}_c Q) + 2u_{(a} \left[-\tilde{\nabla}_{b)} \dot{Q} + \frac{1}{3} \theta \tilde{\nabla}_{b)} Q + (\sigma_{b)c} + \eta_{b)dc} \omega^d \right] \tilde{\nabla}^c Q \\ &\quad + \frac{1}{3} h_{ab} [\tilde{\nabla}^c \tilde{\nabla}_c Q - \dot{Q}\theta] - \dot{Q}\sigma_{ab} + \tilde{\nabla}_a \tilde{\nabla}_b Q. \end{aligned} \quad (66)$$

We also write $\hat{T}_{ab} = \hat{G}_{ab}$, and $T_{ab} = G_{ab}$ as general fluids, both with respect to u^a :

$$\hat{G}_{ab} = \hat{\mu}\hat{u}_a\hat{u}_b + \hat{p}\hat{h}_{ab} + 2\hat{q}_{(a}\hat{u}_{b)} + \hat{\pi}_{ab}, \quad (67)$$

$$G_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab}; \quad (68)$$

where $\{\hat{\mu}, \hat{p}, \hat{q}_a, \hat{\pi}_{ab}\}$, and $\{\mu, p, q_a, \pi_{ab}\}$ are the energy density, isotropic pressure, heat flux, and anisotropic pressure of \hat{G}_{ab} and G_{ab} respectively.

We can decompose G_{ab} given by (65) into the fluid variables in (68) by using (66) in the following covariant manner:

$$\mu = u^a u^b G_{ab} = e^{2Q} \hat{\mu} - 3\dot{Q} \left(\dot{Q} - \frac{2}{3}\theta \right) - 2\tilde{\nabla}_a \tilde{\nabla}^a Q + \tilde{\nabla}_a Q \tilde{\nabla}^a Q, \quad (69)$$

$$p = \frac{1}{3} h^{ab} G_{ab} = e^{2Q} \hat{p} + \left(\dot{Q} - \frac{4}{3}\theta \right) \dot{Q} - 2\ddot{Q} + \frac{4}{3} \tilde{\nabla}_a \tilde{\nabla}^a Q - \frac{5}{3} \tilde{\nabla}_a Q \tilde{\nabla}^a Q + 2\dot{u}_c \tilde{\nabla}^c Q,$$

$$q_a = -u^b G_{\langle a \rangle b} = e^Q \hat{q}_a + 2\dot{Q} \left(\tilde{\nabla}_a Q - \dot{u}_a \right) + 2h_a^b \left(\tilde{\nabla}_b Q \right)' + 4\eta_{abc} \omega^b \tilde{\nabla}^c Q \quad (70)$$

$$\pi_{ab} = G_{\langle ab \rangle} = \hat{\pi}_{ab} + 2\dot{Q} \sigma_{ab} - 2\tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} Q - 2\tilde{\nabla}_{\langle a} Q \tilde{\nabla}_{b \rangle} Q. \quad (71)$$

In the energy flux equation we used the identity

$$\tilde{\nabla}_a \dot{Q} = h_a^b \left(\tilde{\nabla}_b Q \right)' - \dot{Q} \dot{u}_a + \frac{1}{3} \theta \tilde{\nabla}_a Q + \sigma_a^b \tilde{\nabla}_b Q + \eta_{abc} \omega^b \tilde{\nabla}^c Q. \quad (72)$$

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